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Robust output regulation for uncertain nonlinear minimum phase systems under unknown control direction ☆

Yizhou Gong [a], Fanglai Zhu [b], Yang Wang [a,*]

[a] School of Information Science and Technology, ShanghaiTech University, Shanghai, China
[b] College of Electronic and Information Engineering, Tongji University, Shanghai, China

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ABSTRACT

This paper investigates the problem of disturbance rejection for SISO uncertain nonlinear minimum phase systems perturbed by an *unmeasurable* external disturbance under the framework of robust output regulation. The model parameters of the systems in question are uncertain, including the control direction. In addition, the external disturbance can be structured or unstructured but bounded. Towards this end, a novel unknown input observer (UIO)-based regulator is developed to cancel the external disturbance, and a switching mechanism with a monitor function is designed to handle the control direction uncertainty. Notable features are that the unstructured external disturbance can be directly estimated and completely rejected by a sliding mode-based observer, and this new scheme can be applied to systems with non-unitary relative degrees under unknown control direction. The boundedness of the closed-loop system and its asymptotic convergence properties are rigorously proved, which is verified by a numerical example.

1. Introduction

The problem of tracking desired references while rejecting disturbances in the presence of model uncertainties, generically known as robust output regulation, has played a central role throughout the history of control theory [1]. It can be found in myriad engineering applications, including active rotor balancing [2], active noise cancellation [3] and active suspensions [4], etc. Although references and disturbances are both commonly interpreted as external signals in the problem of output regulation, the accessibility of these two types of signals is generally different. In practice, the external disturbance to be canceled is more difficult to obtain, especially when facing a time-varying uncertain operating environment. This work particularly focuses on a rather challenging case where the plant in question is a single-input-single-output (SISO) uncertain nonlinear system described by:

$$\begin{aligned} \dot{x} &= f(x,\mu) + g(x,\mu)(u+d), \\ y &= h(x,\mu), \end{aligned} \tag{1}$$

and the disturbance d to be canceled is assumed to be an unmeasured signal generated by exogenous systems. $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ represent the state, the input, and the output of the plant respectively. The vector μ collects the parameter uncertainty ranging over a given compact set $\mathcal{P} \subset \mathbb{R}^p$. To be specific, the model uncertainty exists in vector field $f(x,\mu)$, input map $g(x,\mu)$, and output map $h(x,\mu)$. In this paper, we denote the sign of the multiplicative term $g(x,\mu)$ as the control direction, which is unknown. Besides, we assume $g(x,\mu)$ is bounded and nonzero for all $x \in \mathbb{R}^n$ and $\mu \in \mathcal{P}$. The overall control objective is then to find a control signal u to recover the zero output of the system facing all kinds of aforementioned uncertainties.

When the control directions are known, so far plenty of representative results have been obtained on the robust output regulation problem [5–12]. Remarkable is the internal model (IM)-based method in completely rejecting structured disturbances, even if the information of disturbances is unknown [6,8]. However, consider the unstructured disturbances, it is claimed in [7] that no finite-dimensional robust regulator exists for asymptotic regulation and only approximate or practical regulation can be obtained, which is one of main drawbacks for the IM-based method. Sacrificing the asymptotic regulation, robust control approaches, such as active disturbance rejection control (ADRC) [9], sliding mode-based control [11,12], and disturbance observer (DOB)-based control [13], are able to effectively cancel both structured and unstructured disturbances. Reposing upon the high-gain observer technique, the ADRC and DOB-based control [9,13] can only

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* Corresponding author.
E-mail addresses: gongyzh2022@shanghaitech.edu.cn (Y. Gong), zhufanglai@tongji.edu.cn (F. Zhu), wangyang4@shanghaitech.edu.cn (Y. Wang).

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achieve approximate regulation even if disturbances are structured. The sliding mode-based control [10–12], though achieving exact cancellation of disturbances, requires the system nonlinearity satisfied with some known expressions.

However, when the control directions are unknown, the robust output regulation problem in the presence of structured and unstructured disturbances becomes rather challenging. Nussbaum-type functions have been effectively incorporated in the control design of several proposals in the literature, such as [14–16], but the transient behavior is often unacceptable in practical implementation problems. In [17,18], the system nonlinearity is assumed to be satisfied with some known bounding functions, instead a switching logic by monitoring a certain performance index is designed. Similarly, in [19], by considering linear systems and assuming that other disturbances are norm-bounded, the monitor function is incorporated into ADRC to solve this problem. Recently, rather than resorting to the monitor function, a barrier function-based prescribed performance control method in [20] has solved such a problem without any knowledge of the system nonlinearity or the control direction. However, in [19,20], only practical regulation can be obtained.

In this paper, to the best of the authors' knowledge, our work is the first to *achieve exact cancellation of both structured and unstructured disturbances for systems with no knowledge of the system nonlinearity or the control direction*. Sharing a similar idea of monitoring a certain performance index with [17,18], we incorporate a switching signal into the regulator to overcome the difficulty of unknown control direction. By resorting to a novel interval observer-based UIO combined with the regulator, exact cancellation of both structured and unstructured disturbances is achieved. We show that after finite switching, once the control input u is unsaturated, the output y is asymptotically regulated to zero and the closed-loop system remains bounded. However, we restrict the system in question to be locally input-to-state stable and strongly minimum phase. If the system is input-to-state stable, our result turns out to be semi-global regulation.

The main novelties lie in the following:

1. With the least prior knowledge of the plant and disturbance, the proposed control protocol secures asymptotic regulation, which is robust to the uncertain parameter set $\mathcal{P}$.
2. Without knowledge of the system nonlinearity or the control direction, the structured and unstructured disturbance can be directly reconstructed and completely rejected.

Notations: In this paper, $\|\cdot\|$ represents the Euclidean norm of the matrices or vectors. For any constant matrix or vector $M \in \mathbb{R}^{m\times n}$ ($\mathbb{R}^m$), $M > (\geq,<,\leq)\ 0$ means that *all elements* of M are $> (\geq,<,\leq)\ 0$ respectively. Denote $M^+ = \max\{M,0\}$ and $M^- = \max\{-M,0\}$. Thus, we have $M = M^+ - M^-$ and $|M| = M^+ + M^-$, where $|M|$ stands for a $m\times n$ matrix ($m\times 1$ vector) formed by taking the absolute value of every element of M. In addition, for any square matrix $N \in \mathbb{R}^{n\times n}$, matrix N is Metzler if its off-diagonal components are all non-negative. $\lambda_{\max}(N)$ and $\lambda_{\min}(N)$ are denoted as the maximum and minimum eigenvalues of matrix N. In this paper, the solution of discontinuous differential equations is understood in Filippov's definition [21].

2. Problem formulation

In this section, we first reformulate the disturbance rejection problem such that an interval observer-based UIO can be employed to design an output feedback regulator to steer the output to zero.

Suppose the system (1) in question has a well-defined relative degree r, then through a possible parameter-dependent change of coordinates denoted by $x \mapsto (\eta,\xi)$, it can be transformed into the normal form [22, Theorem 13.1] described by:

$$\begin{aligned}\dot\eta &= f_0(\eta,\xi,\mu)\\ \dot\xi &= A_c\xi + B_c b(\eta,\xi,\mu)u + B_c\left(b(\eta,\xi,\mu)d + a(\eta,\xi,\mu)\right)\\ y &= C_c\xi \end{aligned}\tag{2}$$

where $\eta \in \mathbb{R}^{n-r}$, $\xi := \begin{pmatrix}\xi_1 & \cdots & \xi_r\end{pmatrix}^\top \in \mathbb{R}^r$, $f_0(\eta,\xi,\mu)$, $a(\eta,\xi,\mu)$ and $b(\eta,\xi,\mu)$ are smooth state-dependent nonlinear functions, besides $f_0(0,0,\mu)=0$ and $a(0,0,\mu)=0$ while the pair $\{A_c,B_c,C_c\}$ is

$$A_c = \begin{pmatrix}\mathbf{0}_{(r-1)\times 1} & \mathbf{I}_{(r-1)\times(r-1)}\\ 0 & \mathbf{0}_{1\times(r-1)}\end{pmatrix},$$
$$B_c = \begin{pmatrix}\mathbf{0}_{1\times(r-1)} & 1\end{pmatrix}^\top,\quad C_c = \begin{pmatrix}1 & \mathbf{0}_{1\times(r-1)}\end{pmatrix}.$$

We make the following assumptions regarding the system (2):

Assumption 2.1. There exist two known positive constants $b_{\min}$, $b_{\max}$ such that

$$0 < b_{\min} \leq |b(\eta,\xi,\mu)| \leq b_{\max}\tag{3}$$

for all $(\eta,\xi)\in\mathbb{R}^n$ and $\mu\in\mathcal{P}$.

Remark 2.1. Note that, $b(\eta,\xi,\mu) = L_g L_f^{r-1}h(x,\mu)$ represents the high-frequency gain of the plant (2). In this paper, without the prior knowledge of its sign, we assume it is bounded away from zero by some known values $b_{\min}$, $b_{\max}$ and cannot change its sign for all $(\eta,\xi)\in\mathbb{R}^n$.

Assumption 2.2. Let $l > 0$, $0 < \rho < \frac{1}{2}$ be any positive constants, assume the unknown disturbance is norm-bounded satisfying $|d(t)| \leq \rho l$. Consider system (2), there exists a differentiable continuous function V_1 satisfying $\underline{\alpha}_0(\|(\eta,\xi)\|) \leq V_1(t,\eta,\xi) \leq \overline{\alpha}_0(\|(\eta,\xi)\|)$ for some class-$\mathcal{K}$ functions $\underline{\alpha}_0(\cdot),\overline{\alpha}_0(\cdot)$ such that for all $\mu\in\mathcal{P}$,

$$\dot V_1 \leq -\alpha_0(\|\eta,\xi\|) + \alpha_1(|u+d|)$$

for $|u+d| \leq l$ and $\|(\eta(0),\xi(0))\| \leq l_0$ where $l_0 > 0$ depends on l, $\alpha_0(\cdot),\alpha_1(\cdot)$ are some class-$\mathcal{K}$ functions.

Remark 2.2. Assumption 2.2 implies that the system (2) is locally *input-to-state* stable (ISS). If $l=+\infty$, Assumption 2.2 is reduced to the ISS property and is not restrictive, since there are a large number of robust control methods such as "H_∞ control" [23] and "high gain" stabilization [24] to achieve such stability property. Such a hypothesis is also assumed in rich literature [25–28] focusing on disturbance rejection.

Assumption 2.3. The system $\dot\eta = f_0(\eta,\xi,\mu)$ is ISS with respect to state η and input ξ for all the initial condition $\eta(0)\in\mathbb{R}^{n-r}$.

Remark 2.3. Assumption 2.3 implies that the system (2) is strongly minimum phase, which is a standard assumption in the topic of nonlinear output regulation [6,8,24,29].

To facilitate the controller design, we add and subtract a term $\beta S_q(t)u$ to the right-hand side of the dynamic of ξ_r, then system (2) is rewritten in a compact form:

$$\begin{aligned}\dot\eta &= f_0(\eta,\xi,\mu)\\ \dot\xi &= A_c\xi + B_c\left(\beta S_q u + \Delta(\eta,\xi,u,d,\mu)\right)\\ y &= C_c\xi\end{aligned}\tag{4}$$

where β is a positive constant to be determined later, and $S_q(t)$ is a binary switching signal governed by a scheme that will be designed in the next section, to cycle through the set $\{-1,1\}$. Note that, Δ is the lumped uncertainty in the form of

$$\Delta(\eta,\xi,u,d,\mu) := a(\eta,\xi,\mu) + b(\eta,\xi,\mu)(u+d) - \beta S_q u.\tag{5}$$

One more assumption needs to be made for the initial condition of state ξ in (4) as follows:

Assumption 2.4. There exist two known constant vectors $\overline{\xi}_a(0)$ and $\underline{\xi}_a(0)$ conforming to $\underline{\xi}_a(0) \leq \xi(0) \leq \overline{\xi}_a(0)$ for all $\mu\in\mathcal{P}$.

Remark 2.4. The disturbance here is not necessarily differentiable continuous but only bounded. However, we cannot tolerate an arbitrarily large disturbance, since system (2) is locally ISS with restrictions $\|(\eta(0),\xi(0))\| \le l_0$ on initial conditions and $|u+d| \le l$ on inputs.

Now, concerning the interconnected system (4), the output regulation problem can be formally cast as:

Problem 2.1. Suppose Assumptions 2.1–2.4 hold. Given any positive constant l, design a control law u for system (4) perturbed by disturbance d bounded by $\bar{d} := \rho l$, such that the closed-loop trajectories starting from any initial condition $(\eta(0),\xi(0)) \in \mathcal{X} := \{(\eta(0),\xi(0)) : \|(\eta(0),\xi(0))\| \le l_0\}$ are bounded and the output y of the plant asymptotically converges to zero as time goes to infinity. ◁

For the sake of clarity, hereafter the uncertainty parameter vector μ is ignored when no confusion is caused.

3. Controller design

This section aims to design the control law u that solves Problem 2.1. For simplicity, the time argument has been omitted in the sequel unless necessary. Thanks to the controllability of the matrix pair $\{A_c, B_c\}$, we propose the following certainty-equivalent control law:

$$u = \bar{u}\ \mathrm{Sat}\left[\frac{-K\hat{\xi} - \hat{\Delta}}{\bar{u}\beta S_q}\right] \tag{6}$$

in which $\hat{\xi}$ and $\hat{\Delta}$ stand for the estimates for ξ and Δ. The saturation function is defined as

$$\mathrm{Sat}\,[x] = \begin{cases} x, & \text{if } |x| \le 1 \\ \mathrm{sign}(x), & \text{if } |x| > 1 \end{cases} \tag{7}$$

for any scalar variable x. The control gain $K \in \mathbb{R}^r$ is chosen such that matrix $A_c - B_c K$ is Hurwitz and the selection of positive constant $\bar{u} \in \mathbb{R}$ will be discussed later. β and S_q is first introduced in (4). Fig. 1 depicts the overview of the proposed control architecture whose basic components are composed of the switching scheme updating S_q and the UIO including the estimates of ξ and Δ.

With $\mathcal{X}$ being a fixed compact set of the initial conditions of the system (2), $\mathcal{B}_r$ is defined as a closed ball of radius r such that $\mathcal{X} \subset \mathcal{B}_r$. Thanks to Assumption 2.2, if $\bar{u}$ is designed to be $\bar{u} \le (1-\rho)l$ such that $|u+d| \le l$, then the compact set $\mathcal{B}_r$ is a forward invariant set for system (4) with r depending on l and l_0. In this context, $\hat{\xi}$ and $\hat{\Delta}$ will be given by a high-order sliding mode (HOSM) differentiator and a novel unknown input re-constructor inspired by [30] respectively.

3.1. HOSM differentiator

In what follows, a HOSM differentiator is proposed as a finite-time observer for ξ.

To this end, we first show that the lumped uncertainty Δ is bounded by $|\Delta(\eta,\xi,u,d)| \le \bar{\Delta}^*$ with

$$\bar{\Delta}^* := \sup_{(\eta,\xi)\in\mathcal{B}_r} |a(\eta,\xi)| + (b_{\max}+\beta)\bar{u} + b_{\max}\bar{d}. \tag{8}$$

and from (4) and (8), the following state-dependent upper bound for rth-order derivative of the output signal y satisfies

$$|\dot{\xi}_r| \le \beta\bar{u} + \bar{\Delta}^*. \tag{9}$$

Inspired by [10] and in light of (9), we are ready to propose a HOSM differentiator for ξ as follows,

$$\begin{aligned} \dot{\hat{\xi}}_i &= \hat{\xi}_{i+1} + \hat{v}_i, \quad i = 1,\dots,r-1 \\ \dot{\hat{\xi}}_r &= \beta S_q u + \hat{v}_r, \end{aligned} \tag{10}$$

where $\hat{\xi} := \begin{pmatrix}\hat{\xi}_1 & \cdots & \hat{\xi}_r\end{pmatrix}^\top \in \mathbb{R}^r$, and $\hat{v} := \begin{pmatrix}\hat{v}_1 & \cdots & \hat{v}_r\end{pmatrix}^\top \in \mathbb{R}^r$ is generated by

$$\hat{v}_i = \tau_i \mathcal{L}^{\frac{1}{r+1-i}} \left|\hat{v}_{i-1}\right|^{\frac{r-i}{r+1-i}} \mathrm{sign}(\hat{v}_{i-1}), \quad i = 1,\dots,r \tag{11}$$

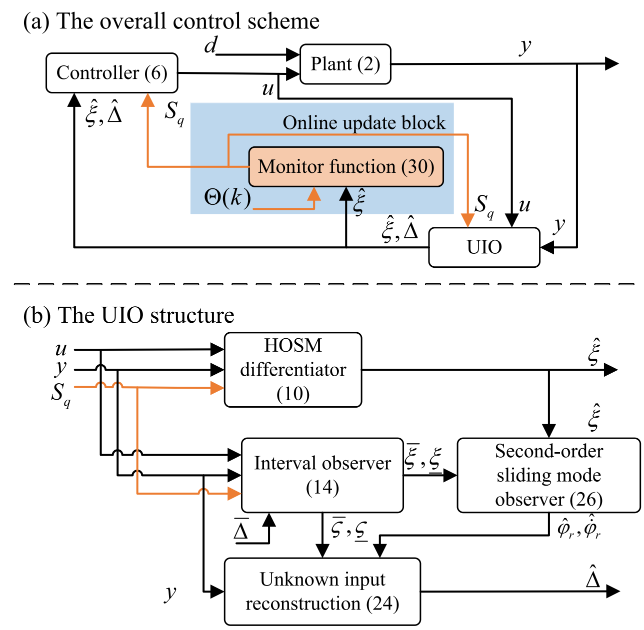

Fig. 1. Schematic of the proposed UIO-based controller with a switching strategy. The blue area in (a) contains the monitor function, driven by the estimate from UIO. The former is described in Section 3.3 while the latter is delineated in Sections 3.1 and 3.2, whose structure is depicted in (b).

and $\hat{v}_0 = y - \hat{\xi}_1$ with $\tau_i, \mathcal{L}$ being positive tuning gains. The finite-time convergence property of $\tilde{\xi} := \hat{\xi} - \xi$ is established in the next lemma.

Lemma 3.1. *Consider the HOSM differentiator (10), if the parameters τ_i are chosen such that the Laplace characteristic polynomial $s^r + \sum_{i=1}^{r} \tau_i s^{i-1}$ is Hurwitz, and static gain $\mathcal{L}$ satisfies*

$$\mathcal{L} \ge \beta\bar{u} + \bar{\Delta}, \tag{12}$$

where $\bar{\Delta}$ is a positive constant satisfying $\bar{\Delta} \ge \bar{\Delta}^*$, then, $\tilde{\xi}$ converges to zero in some finite time $T_1 > 0$. ◁

Proof. In view of (4) and (10), the dynamics of the estimation error $\tilde{\xi}$ can be obtained as

$$\begin{aligned} \dot{\tilde{\xi}}_i &= \tilde{\xi}_{i+1} + \hat{v}_i, \quad i = 1,\dots,r-1 \\ \dot{\tilde{\xi}}_r &\in \hat{v}_r + \left[-\bar{\Delta}, \bar{\Delta}\right], \end{aligned} \tag{13}$$

whose structure is same as that of system (32) in [10]. Thus, appealing to the results indicated in [10, Theorem 5], it can be concluded that the finite time convergence property of $\tilde{\xi}$ is obtained if the static gain $\mathcal{L}$ fulfills $\mathcal{L} \ge \mathcal{L}^* := \beta\bar{u} + \bar{\Delta}$ and parameters τ_i are chosen such that the Laplace characteristic polynomial $s^r + \sum_{i=1}^{r} \tau_i s^{i-1}$ is Hurwitz. □

Remark 3.1. For practical implementation, a proper choice of τ_i can be found in [31,32]. Also, we can always select a sufficiently large $\mathcal{L}$. In addition, the estimation error $\tilde{\xi}$ is bounded during its transient period.

3.2. Interval observer-based estimator

In this subsection, we will develop an interval observer-based estimator for Δ in (5) that features a finite-time convergence property as well.

Thanks to the observability of the matrix pair $\{A_c, C_c\}$, an interval observer for ξ-system (4) is constructed as:

$$\begin{aligned} \bar{\xi}(t) &= (Q^{-1})^{+}\bar{\varsigma}(t) - (Q^{-1})^{-}\underline{\varsigma}(t), \\ \underline{\xi}(t) &= (Q^{-1})^{+}\underline{\varsigma}(t) - (Q^{-1})^{-}\bar{\varsigma}(t), \end{aligned} \tag{14}$$

where the dynamics of $\overline{\varsigma}$ and $\underline{\varsigma}$ are governed by

$$
\begin{aligned}
\dot{\overline{\varsigma}} &= QA_cQ^{-1}\overline{\varsigma} + QB_c\beta S_q u + Q\gamma(y - C_cQ^{-1}\overline{\varsigma}) \\
&+ (QB_c)^{+}\overline{\Delta} - (QB_c)^{-}(-\overline{\Delta}), \\
\dot{\underline{\varsigma}} &= QA_cQ^{-1}\underline{\varsigma} + QB_c\beta S_q u + Q\gamma(y - C_cQ^{-1}\underline{\varsigma}) \\
&+ (QB_c)^{+}(-\overline{\Delta}) - (QB_c)^{-}\overline{\Delta},
\end{aligned}
\tag{15}
$$

in which $Q \in \mathbb{R}^{r\times r}$ and $\gamma \in \mathbb{R}^{r}$ will be determined later, $\overline{\Delta}$ is a positive constant fulfilling $\overline{\Delta} \geq \overline{\Delta}^{*}$ in (12), and the initial conditions are set as $\underline{\varsigma}(0) = Q^{+}\underline{\xi}_a(0) - Q^{-}\overline{\xi}_a(0)$ and $\overline{\varsigma}(0) = Q^{+}\overline{\xi}_a(0) - Q^{-}\underline{\xi}_a(0)$. System (14) is the so-called interval observer whose property is asserted by the following proposition.

Proposition 3.1. *Under Assumption 2.4, the state of system (14) verifies $\underline{\xi} \leq \xi \leq \overline{\xi}$ for all $t \geq 0$, if the gain vector γ together with the matrix Q is chosen such that the matrix $Q(A_c - \gamma C_c)Q^{-1}$ is not only Hurwitz but also Metzler.* ◁

The proof can be found in Appendix A.

Remark 3.2. The construction of an invertible matrix Q and a vector γ such that $Q(A_c - \gamma C_c)Q^{-1}$ is not only Hurwitz but also Metzler can be followed by the procedures in [33]. It can be concluded as follows:

(1) Select a diagonal matrix $R \in \mathbb{R}^{r\times r}$ with its diagonal elements distinct and all negative. γ is obtained such that $(A_c - \gamma C_c)$ and R have the same eigenvalues.
(2) Choose vectors $e_1, e_2 \in \mathbb{R}^{r}$ such that the pairs $(A_c - \gamma C_c, e_1)$ and (R, e_2) are observable. The observable matrices $\mathcal{O}_1$ and $\mathcal{O}_2$ are defined as $\mathcal{O}_1 = \begin{pmatrix} e_1 \\ \vdots \\ e_1(A_c - \gamma C_c)^{r-1} \end{pmatrix}; \mathcal{O}_2 = \begin{pmatrix} e_2 \\ \vdots \\ e_2 R^{r-1} \end{pmatrix}$.
(3) Finally, calculate $Q = \mathcal{O}_2^{-1}\mathcal{O}_1$.

Now, based on the HOSM differentiator (10) and the interval observer (14), we are ready to employ an algebraic unknown input reconstruction method proposed by [30] to estimate the lumped uncertainty Δ in (5). From Proposition 3.1, we have $\underline{\xi} \leq \xi \leq \overline{\xi}$ holds for all $t \geq 0$, which implies $\underline{\xi}_r \leq \xi_r \leq \overline{\xi}_r$, thus there must exist a time-varying scalar $\varphi_r(t) \in [0, 1]$ such that

$$
\xi_r = \varphi_r\overline{\xi}_r + (1 - \varphi_r)\underline{\xi}_r = \varphi_r(\overline{\xi}_r - \underline{\xi}_r) + \underline{\xi}_r. \tag{16}
$$

Then differentiating (16) yields

$$
\dot{\xi}_r = \dot{\varphi}_r(\overline{\xi}_r - \underline{\xi}_r) + \varphi_r(\dot{\overline{\xi}}_r - \dot{\underline{\xi}}_r) + \dot{\underline{\xi}}_r. \tag{17}
$$

Thanks to the fact that $(Q^{-1})^{+} + (Q^{-1})^{-} = |Q^{-1}|$ and in view of (14), it follows that

$$
\begin{aligned}
\overline{\xi} - \underline{\xi} &= \left((Q^{-1})^{+} + (Q^{-1})^{-}\right)\overline{\varsigma} - \left((Q^{-1})^{-} + (Q^{-1})^{+}\right)\underline{\varsigma} \\
&= \left|Q^{-1}\right|\widehat{\varsigma}
\end{aligned}
\tag{18}
$$

with $\widehat{\varsigma} := \overline{\varsigma} - \underline{\varsigma}$ whose dynamic is governed by

$$
\dot{\widehat{\varsigma}} = Q(A_c - \gamma C_c)Q^{-1}\widehat{\varsigma} + \left|QB_c\right|\widehat{\Delta} \tag{19}
$$

where $\widehat{\Delta} := 2\overline{\Delta}$. Further, referring to (14) and (15), it holds that

$$
\begin{aligned}
\dot{\underline{\xi}} &= (Q^{-1})^{+}\dot{\underline{\varsigma}} - (Q^{-1})^{-}\dot{\overline{\varsigma}} \\
&= M_1\underline{\varsigma} - M_2\overline{\varsigma} + \gamma y + N_1(-\overline{\Delta}) - N_2\overline{\Delta} + B_c\beta S_q u
\end{aligned}
$$

where

$$
\begin{aligned}
M_1 &= (Q^{-1})^{+}Q(A_c - \gamma C_c)Q^{-1}, \\
M_2 &= (Q^{-1})^{-}Q(A_c - \gamma C_c)Q^{-1}, \\
N_1 &= (Q^{-1})^{+}(QB_c)^{+} + (Q^{-1})^{-}(QB_c)^{-}, \\
N_2 &= (Q^{-1})^{+}(QB_c)^{-} + (Q^{-1})^{-}(QB_c)^{+}.
\end{aligned}
$$

As a result, in terms of $\dot{\xi}_r$, above equations suggest that

$$
\begin{aligned}
\overline{\xi}_r - \underline{\xi}_r &= f_1(\widehat{\varsigma}), \\
\dot{\overline{\xi}}_r - \dot{\underline{\xi}}_r &= f_2(\widehat{\varsigma}), \\
\dot{\underline{\xi}}_r &= f_3\left(\overline{\varsigma}, \underline{\varsigma}\right) + \beta S_q u
\end{aligned}
\tag{20}
$$

where

$$
\begin{aligned}
f_1(\widehat{\varsigma}) &= B_c^{\top}\left|Q^{-1}\right|\widehat{\varsigma}, \\
f_2(\widehat{\varsigma}) &= B_c^{\top}\left|Q^{-1}\right|\left(Q\left(A_c - \gamma C_c\right)Q^{-1}\widehat{\varsigma} + \left|QB_c\right|\widehat{\Delta}\right), \\
f_3\left(\overline{\varsigma}, \underline{\varsigma}\right) &= B_c^{\top}(M_1\underline{\varsigma} - M_2\overline{\varsigma} + \gamma y + N_1(-\overline{\Delta}) - N_2\overline{\Delta}).
\end{aligned}
\tag{21}
$$

Now, substituting (20) into (17) yields

$$
\dot{\xi}_r = \dot{\varphi}_r f_1(\widehat{\varsigma}) + \varphi_r f_2(\widehat{\varsigma}) + f_3(\overline{\varsigma}, \underline{\varsigma}) + \beta S_q u. \tag{22}
$$

Meanwhile, recalling the second equation of (4), we have $\dot{\xi}_r = \beta S_q u + \Delta$ that, together with (22), gives

$$
\Delta = \dot{\varphi}_r f_1(\widehat{\varsigma}) + \varphi_r f_2(\widehat{\varsigma}) + f_3(\overline{\varsigma}, \underline{\varsigma}). \tag{23}
$$

Now, an unknown input reconstruction of Δ in (5) is given:

$$
\hat{\Delta} = \hat{\dot{\varphi}}_r f_1(\widehat{\varsigma}) + \hat{\varphi}_r f_2(\widehat{\varsigma}) + f_3(\overline{\varsigma}, \underline{\varsigma}) \tag{24}
$$

where $\hat{\dot{\varphi}}_r$ and $\hat{\varphi}_r$ are the estimate of $\dot{\varphi}_r$ and φ_r, respectively. To proceed, due to (16), $\hat{\varphi}_r$ can be computed by

$$
\hat{\varphi}_r = \frac{\hat{\xi}_r - \underline{\xi}_r + \epsilon}{\overline{\xi}_r - \underline{\xi}_r + \epsilon} \tag{25}
$$

with $\epsilon = 1$, if $\overline{\xi}_r = \underline{\xi}_r$; otherwise, $\epsilon = 0$. As for the estimate of $\dot{\varphi}_r$ denoted by $\hat{\dot{\varphi}}_r$, we again resort to Levant's second-order sliding model observer in [10] as follows:

$$
\begin{aligned}
\dot{\rho}_1 &= \iota_1, \quad \iota_1 = -\kappa_1\left|\rho_1 - \hat{\varphi}_r\right|^{\frac{1}{2}}\mathrm{sign}(\rho_1 - \hat{\varphi}_r) + \rho_2, \\
\dot{\rho}_2 &= -\kappa_2\mathrm{sign}(\rho_2 - \iota_1), \\
\hat{\dot{\varphi}}_r &= \rho_2
\end{aligned}
\tag{26}
$$

where two positive scalar gains $\kappa_i, i = 1, 2$ are chosen properly and recursively.

Proposition 3.2. *Suppose Assumption 2.4 holds, the estimator $\hat{\Delta}$ in (24) that consists of (15), (21), (25) and (26) is able to provide an exact estimate for the lumped uncertainty Δ in (5) within a finite time, that is, there exists a time instant $T_2 > 0$ such that the estimation error $\tilde{\Delta} := \hat{\Delta} - \Delta = 0$, for all $t \geq T_2$.* ◁

Proof. From (23) and (24), we can deduce that

$$
\tilde{\Delta} = \tilde{\dot{\varphi}}_r(t) f_1(\widehat{\varsigma}) + \tilde{\varphi}_r(t) f_2(\widehat{\varsigma})
$$

with $\tilde{\dot{\varphi}}_r(t) := \hat{\dot{\varphi}}_r(t) - \dot{\varphi}_r(t)$ and $\tilde{\varphi}_r(t) := \hat{\varphi}_r - \varphi_r$. According to Lemma 3.1, the finite-time convergence property of $\tilde{\dot{\xi}}_r$ implies $\tilde{\varphi}_r = 0$ for all $t \geq T_1$. Since (26) features the same structure with (10), one can easily conclude that, after another period, say t_f, $\hat{\dot{\varphi}}_r$ must equal to $\dot{\varphi}_r$. Then, setting $T_2 := T_1 + t_f$, we have $\tilde{\Delta} = 0$ for all $t \geq T_2$. Moreover, the estimation error $\tilde{\Delta}$ is bounded during its transient period, thus completing the proof. □

3.3. Switching scheme and monitor function design

Before we discuss the switching scheme of S_q, we revert to system (2) and rewrite it for convenience as

$$
\begin{aligned}
\dot{z} &= \mathbf{F}(z) + \mathbf{G}(b(z)u + b(z)d), \\
y &= \mathbf{C}z,
\end{aligned}
\tag{27}
$$

with $z := \begin{pmatrix}\eta^\top & \xi^\top\end{pmatrix}^\top \in \mathbb{R}^n$ and

$$\mathbf{F}(z) = \begin{pmatrix} f_0(z) \\ \begin{pmatrix}\mathbf{0}_{n-r} & A_c\end{pmatrix} z + B_c a(z)\end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix}\mathbf{0}_{n-r} \\ B_c\end{pmatrix},$$
$$\mathbf{C} = \begin{pmatrix}\mathbf{0}_{n-r} & C_c\end{pmatrix}.$$

Since we have chosen an upper boundary for $\bar{u}$ before Section 3.1, i.e., $\bar{u} \le (1-\rho)l$, the finite-time convergence properties of the estimate errors $\tilde{\xi}$ and $\tilde{\Delta}$ are ensured by Lemma 3.1 and Proposition 3.2. However, to recover the asymptotic stability of the unforced system, $\bar{u}$ should be selected such that the ultimate boundary of the unsaturated term $u_0 := \frac{-K\hat{\xi}-\hat{\Delta}}{\beta S_q}$ is smaller than $\bar{u}$. Keeping this in mind, we will state it formally in the following proposition.

Proposition 3.3. *Suppose Assumptions 2.1–2.4 hold, consider any constant l and compact set $\mathcal{B}_r$, the latter being forward invariant for the closed-loop system (27), which satisfies $\mathcal{X} \subset \mathcal{B}_r$. When $S_q = sign(b(z)), \forall z \in \mathcal{B}_r$ and select $\beta \ge b_{\max}$, if $l_0 \le c(l)$ with c being some class-$\mathcal{K}$ function such that there exists a $\bar{u}$ satisfying $\bar{u} \ge \bar{d} + \frac{1}{b_{\min}} \sup_{z\in\mathcal{B}_r} \bar{\alpha}(\|z\|)$, in which $\bar{\alpha}(\|z\|) := |K\xi + a(z)|$, then there exists a time instant T^* such that the unsaturated term $u_0 = \frac{-K\hat{\xi}-\hat{\Delta}}{\beta S_q}$ is norm-bounded by $|u_0| \le \bar{u}$ after $t \ge T^*$, that is $u = u_0$, $\forall t \ge T^*$.* ◁

The proof can be found in Appendix B.

Remark 3.3. If $l = +\infty$, system (2) is ISS with respect to the inputs u, d and states η, ξ. The upper boundary and lower boundary of $\bar{u}$ are naturally valid. Then, our results turn out to be semi-global regulation of the regulated output.

Finally, a switching scheme is proposed as

$$S_q(t_{k+1}) = -S_q(t_k),$$
$$t_{k+1} = \inf\left\{t > t_k : \|\hat{\xi}(t)\| = \Psi_k(t)\right\}, \tag{28}$$

where the switching instant t_k, $k \in \{0,1,2,\dots\}$ determines the change of S_q, at which S_q cycles through the set $\{-1,1\}$. The switching scheme is initialized at $t_0 = 0$. A sequence of signals $\Psi_k(t)$ for all $k \in \{0,1,2,\dots\}$ is defined as

$$\Psi_k(t) = \Theta(k) e^{-\lambda_m(t-t_k)} \|\hat{\xi}(t_k)\|, \tag{29}$$

where λ_m is a sufficiently small positive constant to be determined later and $\Theta(k)$ is a user-defined non-decreasing function of the augment k satisfying $\Theta(k) > 1$. The monitor function Ψ can thus be defined as

$$\Psi(t) := \Psi_k(t), \quad t \in \left[t_k, t_{k+1}\right) \subset [0, +\infty). \tag{30}$$

Note that, from (29) and (30), one can easily see $\|\hat{\xi}(t_k)\| < \Psi_k(t_k)$ at $t = t_k$ for all $k \in \{0,1,2,\dots\}$. Moreover, it is trivial to obtain the following inequality:

$$\|\hat{\xi}(t)\| \le \Psi(t) \tag{31}$$

for all $t \ge 0$.

Remark 3.4. In fact, we can choose a non-decreasing function with a fast ramping rate to substantially reduce the number of switching. If $\Theta(k)$ exceeds the finite accuracy, then a function with a linear ramping rate can be selected, with a tradeoff for a larger switching number.

4. Stability analysis

In this section, consider the system (1) that can be transformed into its normal form (2), its compact form is rewritten in (27). Then, with the controller (6), the entire closed-loop system is in the form of

$$\dot{z} = \mathbf{F}(z) + \mathbf{G}\left(b(z)\bar{u}\,\mathrm{Sat}\left[\frac{-K\hat{\xi}-\hat{\Delta}}{\bar{u}\beta S_q}\right] + b(z)d\right),$$
$$y = \mathbf{C}z, \tag{32}$$

where $\hat{\xi}$ and $\hat{\Delta}$ are given by the finite-time estimators (10) and (24) and S_q is generated by the switching scheme in (28).

Now, we proceed with the stability analysis of the closed-loop system (32), and the main result is shown in the following:

Theorem 4.1. *Suppose Assumptions 2.1–2.4 hold. Given any positive constant l, there exist choices of $\bar{u}$, l_0, β, $\mathcal{L}^*$, $\overline{\Delta}^*$ and λ_m^* such that for all $l_0 \le c(l)$, $\beta \ge b_{\max}$, $\mathcal{L} \ge \mathcal{L}^*$, $\overline{\Delta} \ge \overline{\Delta}^*$ in (12) and $\lambda_m \le \lambda_m^*$ in (29), the closed-loop trajectories (32) with any initial condition $z(0) \in \mathcal{X}$ are all bounded and $\lim_{t\to\infty} y(t) = 0$, thus the problem of output regulation is solved.* ◁

Proof. According to Assumption 2.2, state z of the closed-loop system is bounded within $\mathcal{B}_r$ since $\bar{u}$ satisfies $\bar{u} \le (1-\rho)l$. As a consequence, as long as $\mathcal{L}$ is sufficiently large, then in virtue of Lemma 3.1 and Proposition 3.2, it holds that the estimation errors $\tilde{\xi}$ and $\tilde{\Delta}$ equal to zero after some time T' and during their transient period, $\tilde{\xi}$ and $\tilde{\Delta}$ are bounded, which implies the boundedness of the estimates $\hat{\xi}$ and $\hat{\Delta}$.

Next, we are ready to prove the switching of S_q in (28) stops by contradiction. Suppose the switching does not stop, that is $k \to +\infty$. Thus, there exists either an infinite odd or infinite even sequence Ξ such that for all $j \in \Xi$, $S_q(t)$ coincides with the sign of $b(z)$ during the time interval $t \in [t_j, t_{j+1})$. From Proposition 3.3, there must exist another time instant $T \ge T'$ and a positive constant j' such that for all $j \ge j'$ and $t_{j'} \ge T$, it holds $|u_0| \le \bar{u}$ as well as $\tilde{\xi} = 0$ and $\tilde{\Delta} = 0, \forall t \in [t_j, t_{j+1})$. Hence, $\forall j \ge j'$, during the time interval $t \in [t_j, t_{j+1})$, the closed-loop system becomes

$$\dot{z} = \bar{\mathbf{F}}(z) \tag{33}$$

with

$$\bar{\mathbf{F}}(z) = \begin{pmatrix} f_0(z) \\ \begin{pmatrix}\mathbf{0}_{\mathbf{n-r}} & A_c - B_c K\end{pmatrix} z\end{pmatrix}.$$

Recalling $z = \begin{pmatrix}\eta^\top & \xi^\top\end{pmatrix}^\top$, from (33) one is able to obtain the dynamic of ξ as $\dot{\xi} = (A_c - B_c K)\xi$. For this dynamic equation, we introduce a Lyapunov candidate function $V_2 = \sqrt{\xi^\top P_\xi \xi}$, $\forall t \in [t_j, t_{j+1})$, $j \ge j'$, where P_ξ is a symmetric and positive definite solution of the Lyapunov equation $(A_c - B_c K)^\top P_\xi + P_\xi (A_c - B_c K) = -\mathbf{I}_r$. Then, the time derivative of V_2 along the dynamic of ξ satisfies

$$\dot{V}_2 \le -\frac{1}{2\lambda_{\max}(P_\xi)} V_2, \quad \forall t \in [t_j, t_{j+1}),\ j \ge j'$$

Next, utilizing the comparison lemma [22, Lemma 3.4], it follows that

$$\|\xi(t)\| \le \Sigma(t), \quad \forall t \in [t_j, t_{j+1}),\ j \ge j' \tag{34}$$

with

$$\Sigma(t) := \frac{\lambda_{\max}(P_\xi)}{\lambda_{\min}(P_\xi)} e^{-\lambda_m^*(t-t_j)} \|\xi(t_j)\|$$

and $\lambda_m^* := \frac{1}{2\lambda_{\max}(P_\xi)}$.

When $j \to +\infty$, there must exist a positive constant j^* satisfying $j^* \ge j'$ such that the inequality

$$\Theta(j^*)\|\hat{\xi}(t_{j^*})\| = \Theta(j^*)\|\xi(t_{j^*})\| > \frac{\lambda_{\max}(P_\xi)}{\lambda_{\min}(P_\xi)}\|\xi(t_{j^*})\|$$

holds since $\tilde{\xi} = 0$ after $t \ge T$. Thus, $\Psi_{j^*}(t) > \Sigma(t) \ge \|\hat{\xi}(t)\|$, for all $t \in [t_{j^*}, t_{j^*+1})$. At the time instant $t = t_{j^*+1}$, we have $\Psi_{j^*}(t_{j^*+1}) > \|\hat{\xi}(t_{j^*+1})\|$, which indicates that no switching will occur. This leads to a contradiction. Therefore, the above consideration yields S_q in (28) has to stop switching after $t \in [t_{j^*}, +\infty)$.

After the switching stops, the monitor function Ψ convergences to zero exponentially, guaranteeing the trajectory of ξ converges to zero as well. Under Assumption 2.3 and the fact that ξ converges to zero as $t \to +\infty$, one concludes that η converges to zero asymptotically. Furthermore, with $y = C_c\xi$, one derives $\lim_{t\to\infty} y(t) = 0$, which ends the proof. □

5. Simulation

In this section, we conduct a numerical experiment to validate the performance and robustness of the proposed control protocol in the presence of parameter uncertainties and external disturbances.

Consider a nonlinear minimum phase system described by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -3x_1 - \mu_1 x_2 + b(x)u + \left(\mu_2 \sin(x_1) + b(x)\right) d, \\ y &= x_1 \end{aligned} \tag{35}$$

in which

$$b(x) = \begin{cases} -1 - \frac{x_2^2}{1+x_2^2}, & \text{if } t \in [0, 30) \\ 1 + \frac{x_2^2}{1+x_2^2} & \text{if } t \geq 30 \end{cases}$$

is the high-frequency gain function satisfying $|b(x)| \in [1, 2]$. The unknown parameter vector $\mu = \begin{pmatrix} \mu_1 & \mu_2 \end{pmatrix}^\top$ is assumed to satisfy

$$\mu \in \left\{3.5 \leq \mu_1 \leq 4.5, -1 \leq \mu_2 \leq 1\right\}.$$

The external disturbance is expressed as

$$d(t) = \begin{cases} 5\,\mathrm{sawtooth}(2\pi t), & \text{if } t \in [0, 15) \\ 5\sin(t-1), & \text{if } t \geq 15 \end{cases}$$

where "sawtooth(t)" is a MATLAB function that can generate a sawtooth wave with period 2π for the elements of time t. $d(t)$ is norm-bounded by a constant $\bar{d} = 5$ that is assumed to be known in prior. Let $d_a := \frac{\mu_2 \sin(x_1) + b(x)}{b(x)} d$. The dynamics of x_2 can be rewritten into $\dot{x}_2 = -3x_1 - \mu_1 x_2 + b(x)(u + d_a)$.

Given only the output signal y is available for controller design and the initial condition of the plant (35) ranges over a known compact set $\left\{x \in \mathbb{R}^2 : |x_i(0)| \leq 30, i = 1, 2\right\}$, a streamlined procedure to construct the proposed observer-based controller with the switching mechanism is as follows:

First, the parameter β needed in the certainty-equivalent control law (6) is set as $\beta = 2$ whereas the feedback gain vector $K = \begin{pmatrix} 3 & 4 \end{pmatrix}$ is chosen to place the eigenvalue of $A_c - B_c K$ in (2) to the left-half plane, here for instance, $-1, -3$. Next, to facilitate a fast convergence of $\hat{\xi}$, a sufficiently large $\mathcal{L}$ in the HOSM differentiator (10) and other the tuning gains in (26) are taken according to [31] as $\mathcal{L} = 20, \tau_1 = 1.5, \tau_2 = 1.1$ and $\kappa_1 = 2.12, \kappa_2 = 2.2$. Also, a sufficiently large $\bar{\Delta}$ is chosen to be $\bar{\Delta} = 20$. The gain vector γ and the transformation matrix Q for the interval observer in (15) are selected according to [33] as $\gamma = \begin{pmatrix} 17 & 52 \end{pmatrix}^\top$ and $Q = \begin{pmatrix} -0.4444 & 0.1111 \\ 1.4444 & -0.1111 \end{pmatrix}$ so that $Q(A_c - \gamma C_c)Q^{-1} = \begin{pmatrix} -4 & 0 \\ 0 & -13 \end{pmatrix}$ is a Hurwitz and Metzler matrix, thus satisfying the condition in Proposition 3.1. Finally, for the monitor function Ψ (30) of the switching mechanism (28), we set $\Theta(k) = k + 2$ that is monotonically increasing and $\Theta(k) > 1$ for all $k \in \{0, 1, 2, \ldots\}$ while a sufficiently small λ_m is set as $\lambda_m = 0.5$.

Now, the only parameter unsettled is the saturation constant $\bar{u}$ which needs to be carefully chosen according to the inequalities stated in Proposition 3.3. Since system (35) is ISS with respect to the inputs u, d and state x, the upper boundary of $\bar{u}$ satisfies $\bar{u} \leq (1-\rho)l = +\infty$. Reminiscent of the compact set of the initial condition aforementioned, we take the compact set $\mathcal{B}_r = \{x : \|x\| \leq r = 70\}$ for system (35). Then, the lower boundary of $\bar{u}$ can be computed according to Proposition 3.3 as $\bar{u} \geq 10 + \sup_{x \in \mathcal{B}_r} |(4 - \mu_1)x_2| = 45$. Here, we take the saturation value as $\bar{u} = 60$.

Based on the preceding parameter design, the controller (6) is given by

$$u = 60\,\mathrm{Sat}\left[\frac{-K\hat{\xi} - \hat{\Delta}}{90 S_q}\right]$$

in which $\hat{\xi}$ and $\hat{\Delta}$ are provided by (10) and (24), and S_q is given by (28). The HOSM differentiators in (10) and (26) are initialized with $\hat{\xi}(0) = \begin{pmatrix} 0 & 0 \end{pmatrix}^\top$, $\rho_1(0) = 0.5$, $\rho_2(0) = 0$. The initial values of interval observer (15) are $\bar{\xi}(0) = \begin{pmatrix} 1 & 8 \end{pmatrix}^\top$ and $\underline{\xi}(0) = \begin{pmatrix} -12 & 0 \end{pmatrix}^\top$. The switching signal S_q is initialized as $S_q(0) = 1$ which is opposite to the sign of $b(x)$ in the beginning. This makes our simulation quite stringent.

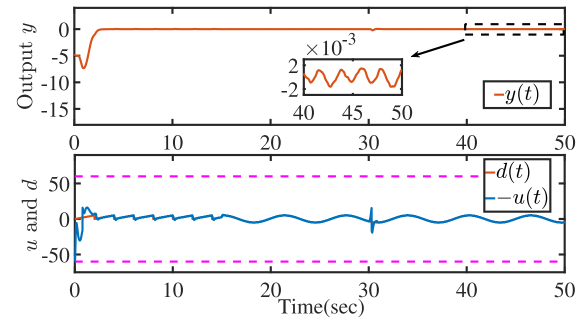

Fig. 2. Trajectories of inputs and output of system (35) with $\bar{u} = 60$.

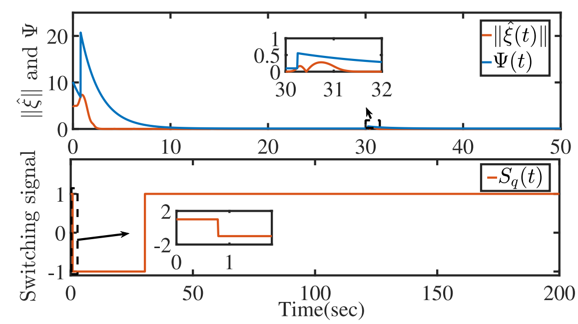

Fig. 3. Monitor function Ψ and switching signal S_q with $\bar{u} = 60$.

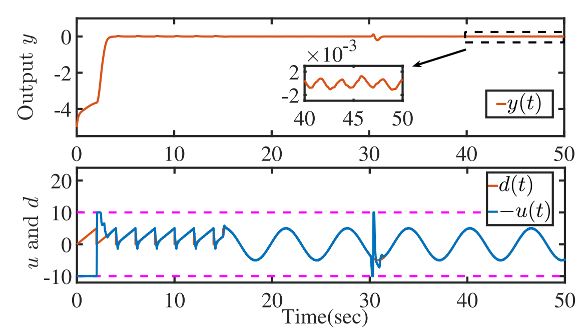

Fig. 4. Trajectories of inputs and output of system (35) with $\bar{u} = 10$.

The results are depicted in Figs. 2–3. Regardless of the sudden change of the structure of the external disturbance d at 15 s, and the change of sign of $b(x)$ at 30 s, the output y of plant (35) has been regulated to zero in a rather short time, along with a small overshoot. It is seen in Fig. 2 that the control input u reconstructs the unstructured external disturbance d very quickly, which is a remarkable feature of the developed method. To be notable, Fig. 3 shows that at 30 s, the switching signal S_q goes through only one change and remains the same, which demonstrates the robustness of our approach concerning the variation of control direction.

The saturation constant $\bar{u}$ utilized in the preceding simulation is chosen according to Proposition 3.3. It might provide a rather conservative feasible range for the choice of $\bar{u}$, which is calculated for all initial conditions and all μ in the given compact sets. For example, Figs. 4–5 show that the closed-loop response of system (35) for $\bar{u} = 10$ still pertains to satisfactory transient performance, in which no frequent switches occur and the regulated output converges to zero very quickly.

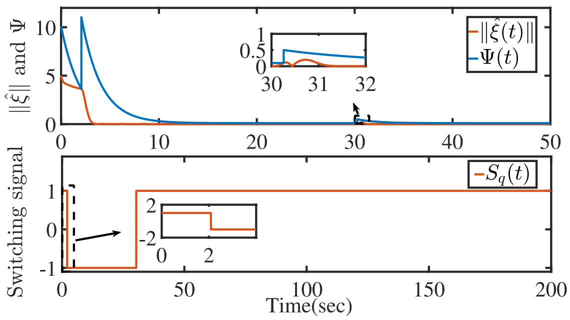

Fig. 5. Monitor function Ψ and switching signal S_q with $\bar{u} = 10$.

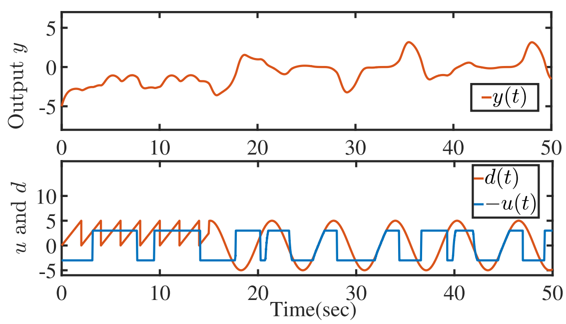

Fig. 6. Trajectories of inputs and output of system (35) with $\bar{u} = 3$.

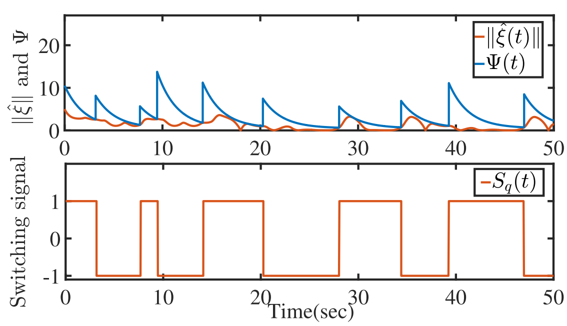

Fig. 7. Monitor function Ψ and switching signal S_q with $\bar{u} = 3$.

If we keep decreasing the value of $\bar{u}$, the asymptotic regulation will not be achieved and only boundedness of trajectories is guaranteed, as depicted in Figs. 6–7 for a more limited choice of $\bar{u} = 3$. In summary, Figs. 2–7 demonstrate how the closed-loop response of system (35) can be influenced by the choice of $\bar{u}$ and further, for a practical u, the admissible range of $\bar{u}$ is much larger than the theoretical range given by Proposition 3.3.

6. Conclusions

In this paper, a novel UIO-based regulator is proposed to solve the output regulation problem for uncertain SISO nonlinear minimum phase systems with arbitrary relative degrees. The usual assumption of the sign of the high-frequency gain known as prior is not required. A finite-time estimator for the parametric uncertainties and external disturbances in the plant can be implemented from a novel unknown input observer. The combination of the unknown input observer and a switching scheme driven by a monitor function allows us to develop a robust and efficient control protocol. It is shown that the external disturbance, structured or unstructured, both can be reconstructed and rejected in a finite time. The closed-loop system will be asymptotically regulated to zero, which is theoretically proven and numerically verified. We would like to draw the reader's attention to the remarkable transient behavior of the system which is mainly due to the use of a series of HOSM-based observers. The authors realize the potential chattering phenomena and other limitations of such kind of solution. Nevertheless, in the case of a structured external disturbance, since the controller is able to completely recover the disturbance signal $d(t)$ to be rejected, one can employ many existing identification techniques to develop an internal model. Thus, in future work, we tend to embed such an internal model into the proposed framework to further improve the performance of the controller. Besides, how to relax Assumption 2.2 will be our future direction.

CRediT authorship contribution statement

Yizhou Gong: Investigation, Methodology, Validation, Writing – original draft. **Fanglai Zhu:** Writing – review & editing. **Yang Wang:** Conceptualization, Methodology, Resources, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A. Proof of Proposition 3.1

To this end, two necessary lemmas are introduced first:

Lemma A.1 ([34]). *Suppose that the matrix $A \in \mathbb{R}^{n\times n}$ is a Metzler and Hurwitz matrix, besides $d_x(t) \in \mathbb{R}^n$, $d_x(t) \geq 0$, $t \geq 0$, then the solution of the dynamics $\dot{x}(t) = Ax(t) + d_x(t)$ satisfies $x(t) \geq 0$ for all $t \geq 0$ if $x(0) \geq 0$.*

Lemma A.2 ([34]). *Suppose that the vector variables $\underline{x}(t) \in \mathbb{R}^n$, $x(t) \in \mathbb{R}^n$ and $\overline{x}(t) \in \mathbb{R}^n$ satisfies $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$, then for any constant matrix $M \in \mathbb{R}^{m\times n}$, we have $M^+\underline{x}(t) - M^-\overline{x}(t) \leq Mx(t) \leq M^+\overline{x}(t) - M^-\underline{x}(t)$.*

Consider the coordinate change $\varsigma := Q\xi$ with the invertible transformation matrix Q defined in (15), the dynamics of ξ in (4) is transformed into

$$\dot{\varsigma} = QA_cQ^{-1}\varsigma + QB_c\beta S_q u + QB_c\Delta(\eta, \xi, u, d)$$
$$y = C_cQ^{-1}\varsigma.$$

Define $\overline{\varsigma}_e := \overline{\varsigma} - \varsigma$ and $\underline{\varsigma}_e := \varsigma - \underline{\varsigma}$. From (15) and the dynamics of ς, it follows that

$$\dot{\overline{\varsigma}}_e = Q(A_c - \gamma C_c)Q^{-1}\overline{\varsigma}_e + \delta_1,$$
$$\dot{\underline{\varsigma}}_e = Q(A_c - \gamma C_c)Q^{-1}\underline{\varsigma}_e + \delta_2,$$

with

$$\delta_1 := (QB_c)^+\overline{\Delta} - (QB_c)^-(-\overline{\Delta}) - QB_c\Delta(\eta, \xi, u, d),$$
$$\delta_2 := QB_c\Delta(\eta, \xi, u, d) - (QB_c)^+(-\overline{\Delta}) + (QB_c)^-\overline{\Delta}.$$

Then, by the selection of $\overline{\Delta}$ in (12) and according to Lemma A.2, we obtain

$$\delta_1 = (QB_c)^+\overline{\Delta} - (QB_c)^-(-\overline{\Delta}) - QB_c\Delta(\eta, \xi, u, d) \geq 0,$$
$$\delta_2 = QB_c\Delta(\eta, \xi, u, d) - (QB_c)^+(-\overline{\Delta}) + (QB_c)^-\overline{\Delta} \geq 0.$$

Under Assumption 2.4 that $\underline{\xi}_a(0) \leq \xi(0) \leq \overline{\xi}_a(0)$ and by $\varsigma(0) = Q\xi(0)$, we derive $Q^+\underline{\xi}_a(0) - Q^-\overline{\xi}_a(0) \leq Q\xi(0) \leq Q^+\overline{\xi}_a(0) - Q^-\underline{\xi}_a(0)$. Recalling $\underline{\varsigma}(0) = Q^+\underline{\xi}_a(0) - Q^-\overline{\xi}_a(0)$ and $\overline{\varsigma}(0) = Q^+\overline{\xi}_a(0) - Q^-\underline{\xi}_a(0)$, thus it follows that $\underline{\varsigma}(0) \leq \varsigma(0) \leq \overline{\varsigma}(0)$, that is $\underline{\varsigma}_e(0) \geq 0, \overline{\varsigma}_e(0) \geq 0$. Therefore, based on Lemma A.1, from the dynamics of $\underline{\varsigma}_e, \overline{\varsigma}_e$, if $Q(A_c - \gamma C_c)Q^{-1}$ is a Metzler and Hurwitz matrix, we conclude that $\underline{\varsigma}_e \geq 0$ and $\overline{\varsigma}_e \geq 0$ for all $t \geq 0$. Thus, $\underline{\varsigma} \leq \varsigma \leq \overline{\varsigma}$ for all $t \geq 0$. Finally, employing the relation $\xi = Q^{-1}\varsigma$ and Lemma A.2, it follows that $\underline{\xi} \leq \xi \leq \overline{\xi}$ for all $t \geq 0$ with $\underline{\xi}, \overline{\xi}$ defined in (14).

Appendix B. Proof of Proposition 3.3

Under Assumption 2.2 that $|u + d| \leq l$ and $|d| \leq \rho l$, we require the upper boundary of $\overline{u}$ satisfying $\overline{u} \leq (1-\rho)l$. Besides, the lower boundary of $\overline{u}$ should fulfill $\overline{u} \geq \overline{d} + \frac{1}{b_{\min}} \sup_{z \in \mathcal{B}_r} \overline{\alpha}(\|z\|)$, which is always valid under the restriction $\|(\eta(0), \xi(0))\| \leq l_0$ on the initial condition. To be clear:

With the locally ISS property under Assumption 2.2, there exist some class-$\mathcal{K}$ functions $c_0(\cdot), c_1(\cdot), c_2(\cdot)$, such that $\sup_{z \in \mathcal{B}_r} \overline{\alpha}(\|z\|) = c_0(r) \leq c_1(l)c_2(l_0)$. Then if l_0 satisfies

$$l_0 \leq c_2^{-1}\left(\frac{b_{\min} c_3(l)}{c_1(l)}\right)$$

with $c_3(l)$ defined as $c_3(l) = (1 - 2\rho)l$, we obtain

$$\frac{1}{b_{\min}} \sup_{z \in \mathcal{B}_r} \overline{\alpha}(\|z\|) \leq \frac{1}{b_{\min}} c_1(l)c_2(l_0) \leq c_3(l),$$

which implies $\overline{d} + \frac{1}{b_{\min}} \sup_{z \in \mathcal{B}_r} \overline{\alpha}(\|z\|) \leq (1-\rho)l$. Thus, the upper boundary and lower boundary of $\overline{u}$ are always valid with the sacrifice of a small compact set for the initial conditions.

To prove u_0 bounded by $|u_0| \leq \overline{u}$, reminiscent of the definition of u_0 and Δ in (5), we replace $\hat{\xi}$ and $\hat{\Delta}$ with ξ and Δ since $\tilde{\xi} = 0$ and $\tilde{\Delta} = 0$ after $T \geq T_2$. Define $\beta_q = \beta S_q$ for the sake of clarity and recall $\overline{\alpha}(\|z\|) := |K\xi + a(z)|$. Utilizing $\overline{u} \geq \overline{d} + \frac{1}{b_{\min}} \sup_{z \in \mathcal{B}_r} \overline{\alpha}(\|z\|)$, after $t \geq T_2$, we have

$$\begin{aligned}
|u_0| &\leq |\frac{K\xi + a(z)}{\beta_q}| + |\frac{b(z) - \beta_q}{\beta_q}|\overline{u} + |\frac{b(z)}{\beta_q}|\overline{d} \\
&\leq |\frac{1}{\beta_q}| \sup_{z \in \mathcal{B}_r} \overline{\alpha}(\|z\|) + |\frac{b(z)}{\beta_q}|\overline{d} - \left(\frac{|\beta_q| - |b(z) - \beta_q|}{|\beta_q|}\right)\overline{u} + \overline{u} \\
&= |\frac{1}{\beta}| \sup_{z \in \mathcal{B}_r} \overline{\alpha}(\|z\|) + |\frac{b(z)}{\beta_q}|\overline{d} - \frac{|b(z)|}{|\beta_q|}\overline{u} + \overline{u} \\
&\leq \left(1 - \frac{|b(z)|}{b_{\min}}\right)|\frac{\overline{u}}{\beta_q}| + \overline{u} \leq \overline{u}
\end{aligned}$$

where we exploit $|\beta_q| - |b(z) - \beta_q| = |b(z)|$ if $S_q = \mathrm{sign}(b(z)), \forall z \in \mathcal{B}_r$ and $\beta \geq b_{\max}$.

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